# Worksheet 2

Name: SOL'NS

Score: ________

Consider two bases, $\mathcal{B}$ and $\mathcal{C}$ for $\mathbb{R}^2$ or $\mathbb{R}^3$ and a vector $[\vec{x}]_{\mathcal{B}}$ in $\mathcal{B}$ coordinates. Find the change of basis matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ and rewrite the vector in $\mathcal{C}$ coordinates.

to $\mathcal{C}$ from $\mathcal{B}$

$$\left[\begin{array}{cc|cc} -2 & -1 & 3/2 & -8/2 \\ -2 & -3 & 1 & -12 \end{array}\right]$$

$$\cancel{\left[\begin{array}{cc|cc} 1 & 1/2 & 3/2 & -4 \\ 0 & -2 & 4 & -28 \end{array}\right]}$$

1. $\mathcal{C} = \left\{ \begin{bmatrix} -2 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \end{bmatrix} \right\}$ $\quad \mathcal{B} = \left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -8 \\ -12 \end{bmatrix} \right\}$

   $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} -2 \\ 13 \end{bmatrix}$ $\quad P = \begin{bmatrix} -2 & 3 \\ 1 & 2 \end{bmatrix}$

$-2 \quad 10$

$$\cancel{\left[\begin{array}{cc|cc} 1 & 0 & 5/2 & 1 \\ 0 & 1 & -2 & 10 \end{array}\right]}$$

2. $\mathcal{C} = \left\{ \begin{bmatrix} 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right\}$ $\quad \mathcal{B} = \left\{ \begin{bmatrix} 4 \\ -9 \end{bmatrix}, \begin{bmatrix} -1 \\ 6 \end{bmatrix} \right\}$

   $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} -17 \\ 20 \end{bmatrix}$ $\quad P = \begin{bmatrix} 3 & -2 \\ -4 & 1 \end{bmatrix}$

$$\left[\begin{array}{cc|cc} -2 & -1 & 3 & -8 \\ 0 & \cancel{2}\ 1 & \cancel{3}\ 1 & \cancel{-4}\ 2 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} -2 & 0 & 4 & -6 \\ 0 & 1 & 1 & 2 \end{array}\right]$$

3. $\mathcal{C} = \left\{ \begin{bmatrix} 4 \\ 0 \end{bmatrix}, \begin{bmatrix} -12 \\ 3 \end{bmatrix} \right\}$ $\quad \mathcal{B} = \left\{ \begin{bmatrix} 16 \\ -3 \end{bmatrix}, \begin{bmatrix} -16 \\ 3 \end{bmatrix} \right\}$

   $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} -23 \\ 14 \end{bmatrix}$ $\quad P = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

$$\left[\begin{array}{cc|cc} 1 & 0 & -2 & 3 \\ 0 & 1 & 1 & 2 \end{array}\right]$$

$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$

4. $\mathcal{C} = \left\{ \begin{bmatrix} -2 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right\}$ $\quad \mathcal{B} = \left\{ \begin{bmatrix} -6 \\ 9 \end{bmatrix}, \begin{bmatrix} 4 \\ -9 \end{bmatrix} \right\}$

   $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} -13 \\ 11 \end{bmatrix}$ $\quad P = \begin{bmatrix} 3 & -2 \\ 3 & 1 \end{bmatrix}$

5.

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \right\} \qquad \mathcal{C} = \left\{ \begin{bmatrix} -3 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 5 \\ 4 \\ 9 \end{bmatrix} \right\}$$

$$[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 0 \\ 4 \end{bmatrix} \qquad P = \begin{bmatrix} -1 & 1 & 2 \\ 2 & -1 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

6. Is it always true that

$$[c.\vec{x}]_{\mathcal{B}} = c\,[\vec{x}]_{\mathcal{B}}$$

and

$$[\vec{x} + \vec{y}]_{\mathcal{B}} = [\vec{x}]_{\mathcal{B}} + [\vec{y}]_{\mathcal{B}}?$$

Yes: the map $\vec{x} \to [\vec{x}]_{\mathcal{B}}$ is linear map assoc to the matrix $P_{\mathcal{B}}^{-1}$

That is, is the process of rewriting vectors in a new coordinate system $\mathcal{B}$ a linear map?

7. Find the change of basis matrix from $\mathcal{B}$ to $\mathcal{C}$ for two bases for the vector space $\mathbb{P}_2$ of polynomials of degree up to 2. 2 ways to solve: use #8 below & write each poly of $\mathcal{B}$ in terms of $\mathcal{C}$, or use $\mathcal{E} = \{1, x, x^2\}$ std basis, figure out $P_{\mathcal{B}}$, $P_{\mathcal{C}}$.

$$\mathcal{B} = \{x^2 + x + 1, x^2 + 1, x - 1\} \qquad \mathcal{C} = \{2x^2 + 3x + 1, 2x^2 + 2x + 1, -x^2 - 2\}$$

Use it to write the polynomial

$$p(x) = 1(x^2 + x + 1) + 2(x^2 + 1) + 3(x - 1)$$

in the new basis $\mathcal{C}$.

$P_{\mathcal{C}}^{-1}\vec{x} = [\vec{x}]_{\mathcal{C}}$, $P_{\mathcal{B}} = [\vec{b}_1\ \vec{b}_2 \cdots \vec{b}_n]$, so $P_{\mathcal{C}}^{-1}P_{\mathcal{B}}$ has col's $[\vec{b}_1]_{\mathcal{C}}$, $[\vec{b}_2]_{\mathcal{C}}$, $[\vec{b}_3]_{\mathcal{C}}$, ...

8. What are the columns of the matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$? Hint: think of the matrix as the composite $P_{\mathcal{C}}^{-1}P_{\mathcal{B}}$. What are the columns of $P_{\mathcal{B}}$? What happens when you apply $P_{\mathcal{C}}^{-1}$ to them?

9. Suppose I want to convert from a basis $\mathcal{A}$ to a basis $\mathcal{C}$ and I already know the matrices

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} \qquad \underset{\mathcal{B}\leftarrow\mathcal{A}}{P}.$$

How do I find $\underset{\mathcal{C}\leftarrow\mathcal{A}}{P}$?

$$\underset{\mathcal{C}\leftarrow\mathcal{A}}{P} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P} \cdot \underset{\mathcal{B}\leftarrow\mathcal{A}}{P}$$

10. Which matrices are change-of-basis matrices $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$? Are all matrices change-of-basis matrices? Only if col's corresp to a basis, i.e., if matrix is invertible.